z-Transform and Its Applications

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⇨ 7.1. INTRODUCTION

The *z*-transform is one of the mathematical tools used for the solution of difference equations. (Mostly discrete-time systems are described by the difference equations). The *z*-transform for discrete-time signals is the counterpart of the Laplace transform for continuous-time signals, and they each have a similar relationship to the corresponding Fourier transform (*i.e.*, continuous-time Fourier transform and discrete-time Fourier transform are the special cases of Laplace transform and *z*-transform respectively). One motivation for introducing this generalization (*z*-transform) is that the discrete-time Fourier transform does not converge for all sequences and it is useful to have a generalization of the discrete-time Fourier transform that encompasses a broader class of signals.

⇨ 7.2. DIFFERENCE EQUATIONS

Just as the differential equations can be used to represent the continuous-time systems $y(t)$, the difference equations can be used to represent the discrete-time systems $y(nt)$ or $y[n]$.

A linear discrete-time system operates in somewhat the same way as continuous-time system, that the present system output $y(nT)$, must be computed using the present input x(nT), past inputs $x(nT - rT)$, and past system outputs $y(nT - mT)$. The general difference equation is in the form of

$$y(nT) + K_1 y(nT - T) + K_2 y(nT - 2T) + \ldots + K_m y(nT - mT)$$
$$= L_0 x(nT) + L_1 x(nT - T) + \ldots + L_r x(nT - rT)$$

⇨ 7.3. BLOCK DIAGRAM REPRESENTATION

An important property of system described by linear constant coefficients difference equations is that they can be represented in very simple and natural ways in terms of block diagram interconnections of elemantry operations.

We begin with the discrete-time case and, in particular, the causal system described by the first-order difference equation.

$$y[n] + a\,y[n-1] = bx[n]$$

To develop a block diagram representation of this system, note that the evaluation of above equation requires three basic operations: addition, multiplication by a coefficient, and dalay (to capture the relationship between $y[n]$ and $y[n-1]$. Thus, let us define three basic network elements as indicated in figure 7.1.

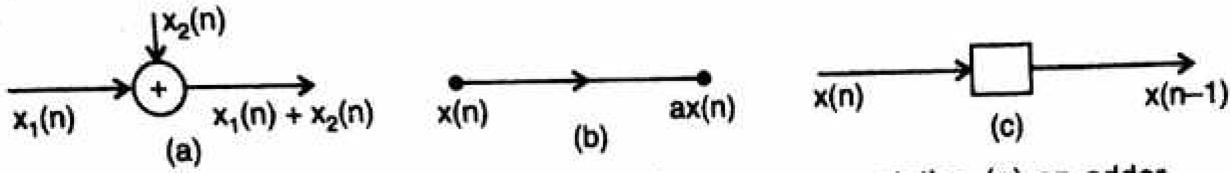

Fig. 7.1. Basic elements for the block diagram representation (a) an adder, (b) multiplication by a coefficient, (c) a unit delay

We rewrite this equation in the form that directly suggests a recursive algorithm for computing successive values of the output $y[n]$:

$$y[n] = -ay[n-1] + bx[n]$$

This algorithm is represented pictorially in figure 7.2.

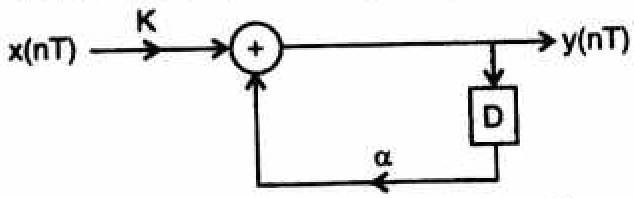

Fig. 7.2. Block diagram representation for the difference equation $y[n] = -ay[n-1] + bx[n]$

EXAMPLE 7.1. Give the block diagram representation of the difference equation.

$$y(nT) = K\,x(nT) + \alpha\,y(nT - T).$$

Solution: Refer to figure 7.3.

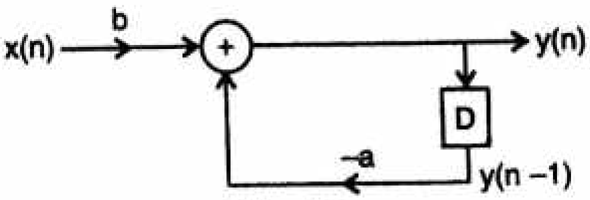

Fig. 7.3.

EXAMPLE 7.2. Solve the given difference equation.

$$y[n] - \frac{1}{2}y[n-1] = x[n]; \quad x[n] = K\,\delta[n]$$

Solution : Since $x[n] = K\,\delta[n]$, *i.e.*, $x[n] = 0$ for $n \neq 0$ (or $n < 0$ and $n > 0$)

Therefore, we started from n = 0,

At $n = 0$, $x[n] = K$ and $y[n] = 0$ for $n < 0$.

$$y[n] = x[n] + \frac{1}{2}y[n-1]$$

$$n = 0 \Rightarrow y[0] = x[0] + \frac{1}{2}y[-1] = K + 0 = K$$

$$y[1] = x[1] + \frac{1}{2}y[0] = 0 + \frac{1}{2}K = \frac{1}{2}K$$

$$y[2] = x[2] + \frac{1}{2}y[1] = 0 + \frac{1}{2}\cdot\frac{1}{2}K = \left(\frac{1}{2}\right)^2 K$$

$$y[n] = \left(\frac{1}{2}\right)^n K\, U[n]$$

⇨ 7.4. THE z-TRANSFORM

7.4.1. Definition of the z-Transform

For a general discrete-time signal $x[n]$, the z-transform, $X(z)$ is defined as

$$X(z) = z\text{-transform of } x[n] = \mathcal{Z}\{x[n]\}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \qquad \text{...(A)}$$

where $n = -\infty$ to ∞ could represent a sequence of numbers or samples, z is a complex variable with real and imaginary parts. One significance of the above definition is that the z-transform converts the sequence of numbers in the real domain into an expression in the complex z-domain.

The variable z is generally complex valued. The z-transform defined in equation (A) is often called the bilateral (or two-sided) z-transform. This pair will be denoted symbolically as

$$x[n] \longleftrightarrow X(z)$$

7.4.2. The Region of Convergence

As in the case of the Laplace transform, the range of values of the complex variable z for which the z-transform $X(z)$ of a discrete-time signal $x[n]$ converges, is called the region of convergence. This concept will become more clear in the following examples:

EXAMPLE 7.3. Determine the z–transform of the signal

$$x[n] = \alpha^n U[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Solution:

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n$$

If $|\alpha z^{-1}| < 1$ or $|z| > |\alpha|$, this power series converges to

$$\frac{1}{1-\alpha z^{-1}}$$

Thus we have

$$X(z) = \frac{1}{1-\alpha z^{-1}}, \qquad |z| > |\alpha|$$

The region of convergence (ROC) is the exterior of a circle having radius $|\alpha|$. Figure 7.4 shows a graph of the signal $x[n]$ and its corresponding ROC.

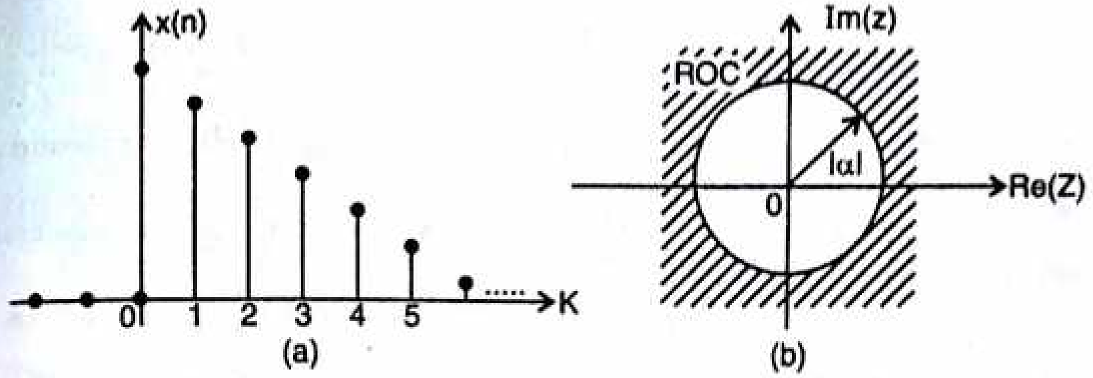

Fig. 7.4. (a) The signal $x[n] = a^n U[n]$, and (b) The ROC of its z-transform

EXAMPLE 7.4. Determine the z-transform of the signal

$$x[n] = -\alpha^n U[-n-1] = \begin{cases} 0 \quad , & n \geq 0 \\ -\alpha^n, & n \leq -1 \end{cases}$$

Solution:
$$X(z) = \sum_{n=-\infty}^{-1} (-\alpha^n) z^{-n} = -\sum_{n=-\infty}^{-1} (\alpha^{-1}z)^{-n} = -\sum_{m=1}^{\infty} (\alpha^{-1}z)^m$$

where $m = -n$

$$X(z) = -\frac{\alpha^{-1}z}{1-\alpha^{-1}z} = \frac{1}{1-\alpha z^{-1}}$$

provided that $|\alpha^{-1}z| < 1$ or $|z| < |\alpha|$. Thus the ROC is now the interior of a circle having radius $|\alpha|$. This is shown in figure 7.5.

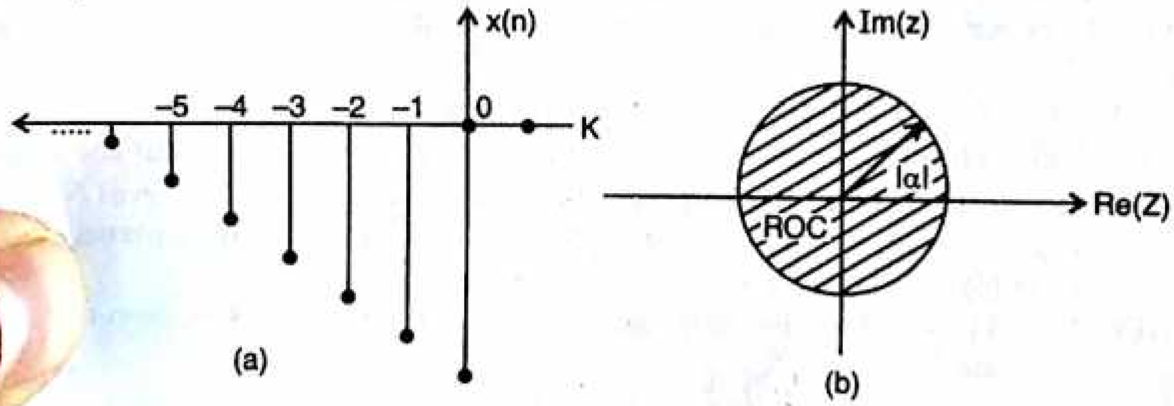

Fig. 7.5. (a) The signal $x[n] = -a^n U[-n-1]$, and (b) The ROC of its z-transform

Note :

Examples 7.3 and 7.4 illustrate two very important issues. The first concerns the uniqueness of the z-transform, we see that the signals $\alpha^n U[n]$ and $-\alpha^n U[-n-1]$ have same expression for the z-transform *i.e.,*

$$\mathcal{Z}[\alpha^n U[n]] = \mathcal{Z}[-\alpha^n U[-n-1]] = \frac{1}{1-\alpha z^{-1}}$$

But their ROCs are complementary to each other, *i.e.,* don't overlap. This implies that an expression for the z-transform does not uniquely specify the signal in time-domain. This problem can be resolved only if an addition to the expression, the ROC is specified. In summary, a discrete-time signal $x[n]$ is uniqvely determined by its z-transform $X(z)$ and the region of convergence (ROC) of $X(z)$.

(VI) If $x[n]$ is two-sided sequence, *i.e.*, neither right-sided nor left-sided, then the ROC is of the from

$$r_1 < \quad |z| < r_2$$

where r_1 and r_2 are the magnitudes of the two poles of $X(z)$. Thus, the ROC is an annular ring in the z-plane between the circles $|z| = r_1$ and $|z| = r_2$ not containing any poles or if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring the z-plane that include the circle $|z| = r_0$.

(VII) If the z-transform $X(z)$ of a discrete-time signal $x[n]$ is rational, then its ROC is bounded by poles or extends to infinity.

7.4.4. Definition of the Unilateral z-Transform

The unilateral or one-sided z-transform $X_1(z)$ of a discrete-time signal $x[n]$ is defined as

$$X_1(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n} \qquad \text{... (3)}$$

It differs form the bilateral z-transform in that the summation is carried over only $n \geq 0$. In other words, the bilateral and unilateral z-transforms are equivalent only if $x[n] = 0$ for $n < 0$ or the unilateral z-transform of $x[n]$ can be thought of as the bilateral z-transform of $x[n]$ multiplied by $U[n]$. The unilateral z-transform is useful for calculating the response of a causal system to a causal input.

7.4.5. Relationship Between the z-Transform and the Laplace Transform

It may be useful to represent a continuous function (signal) $x(t)$ in the form of discrete function (signal) $x(nT)$ or $x(n)$, $n = 0, 1, 2, ...$ as a train of impulse separated by the time interval T as shown in figures 7.7 (*a*) and (*b*). This T is defined as the sampling period. The impulse at th~ :.th time instant, $\delta(t - nT)$, carries the value of $x(nT)$. This situation occurs quite often in digital and sampled–data control system in which a signal $x(t)$ is digitized or sampled every T seconds to form a time sequence that represents the signal at the sampling instants. Thus we can relate the $x(nT)$ with a signal that can be expresed as

$$x^*(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT) \qquad \text{...(2)}$$

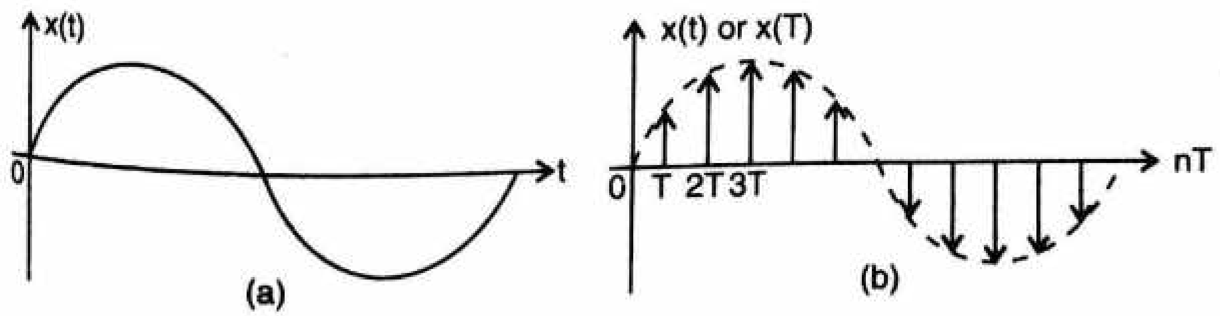

Fig. 7.7.

Taking the Laplace transform on both sides of equation (2), we have

$$X^*(s) = \sum_{n=-\infty}^{\infty} x(nT)\, e^{-nTs} \qquad ...(3)$$

Comparing equation (3) with equation (1), we see that the z-transform may be related to the Laplace transform through

$$z = e^{Ts} \qquad ...(4)$$

with the understanding that the function $x(t)$ is first sampled or discretized (by putting $t = nT$) to get $x^*(t)$ before taking the z-transform.

Thus we can summarize the definition of the z-transform as

$$X(z) = \mathcal{Z}[x(nT)] = \mathcal{Z}[x^*(t)]$$

$$= X^*(s)\big|_{e^{Ts}=z} \qquad ...(5)$$

Recall from Chapter 6 that the Laplace variable s was given by

$$s = \sigma + j\omega$$

where σ was a constant (real axis) to ensure convergence of the integral defining the Laplace transform and thus the existance of the transform itself. From equation (4)

$$z = e^{\sigma T} \cdot e^{j\omega T}$$

So that the magnitude of z is given by

$$|z| = e^{\sigma T}$$

Thus, the right half s-plane, $\sigma > 0$, corresponds to $|z| > 1$, while the left-half s-plane, $\sigma < 0$, corresponds to $|z| < 1$. We see that the left-half s-plane maps into the interior of the unit circle in the z-plane, the right-half s-plane maps outside the unit circle in the z-plane and $j\omega$ axis in the s-plane maps to unit circle in the z-plane. This mapping of the Laplace variable (s) into the z-plane through $z = e^{sT}$ is illustrated in figure 7.8.

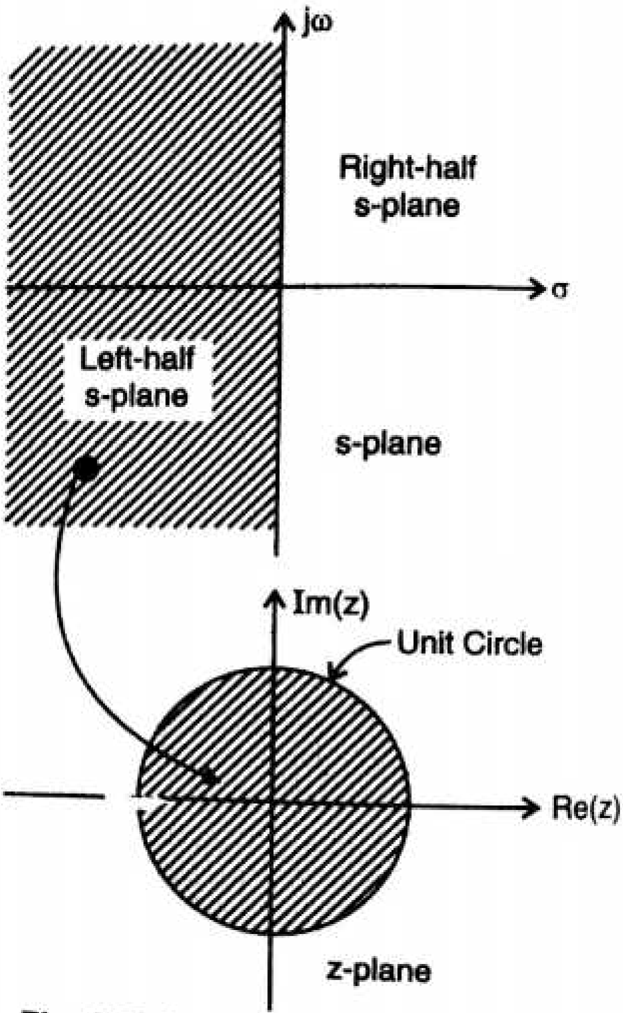

Fig. 7.8. Mapping defined by $z = e^{sT}$

⇨ 7.5. PROPERTIES OF THE z-TRANSFORM

The properties of z-transform are useful in the derivation of z-transforms of many functions and also in the solution of linear constant coefficient difference equations. As with the other transforms, we have discussed, the derivations of the properties of the z-transform are analogous to the derivations of the properties for the other transforms.

7.5.1. Linearity

If $x_1[n]$ and $x_2[n]$ have z-transforms $X_1(z)$ and $X_2(z)$, with their corresponding ROCs R_1 and R_2, respectively, then the z-transform of the linear combination of $x_1[n]$ and $x_2[n]$ is the linear combination of $X_1(z)$ and $X_2(z)$ with ROC is atleast the intersection of R_1 and R_2 (if there is no pole-zero cancellation).

Mathematically,

If $\qquad x_1[n] \xleftrightarrow{z} X_1(z)\,; \quad \text{ROC} = R_1$

and $x_2[n] \xleftrightarrow{z} X_2(z)$; ROC = R_2

Then $a_1 x_1[n] + a_2 x_2[n] \xleftrightarrow{z} a_1 X_1(z) + a_2 X_2[n]$; ROC = $R_1 \cap R_2$

where a_1 and a_2 are arbitrary constants.

If the linear combination is such that some zeros are introduced that cancel poles, then the ROC may be larger as illustrated in following examples.

EXAMPLE 7.6. Calculate the z-transform and its ROC of a discrete-time signal $x[n] = \alpha^n U[n] - \alpha^n U[n-1]$.

Solution: $x[n] = \alpha^n U[n] - \alpha^n U[n-1]$

$$X(z) = \frac{1}{1-\alpha z^{-1}} - \frac{\alpha z^{-1}}{1-\alpha z^{-1}} = \frac{1-\alpha z^{-1}}{1-\alpha z^{-1}} = 1$$

Here, the sequences $\alpha^n U[n]$ and $\alpha^n U[n-1]$ both of infinite duration and their corresponding z-transforms have ROC $|z| > |\alpha|$, while the linear combination is of finite duration, *i.e.*, $\alpha^n U[n] - \alpha^n U[n-1] = \delta[n]$ has a region of convergence that is the entire z-plane.

7.5.2. Multiplication by a Constant

If $x[n] \xleftrightarrow{z} X(z)$; ROC = R

then $\alpha x[n] \xleftrightarrow{z} \alpha X(z)$; ROC = R

(where α is a constant)

7.5.3. Time-Shifting (or Real Translation)

$$x[n] \xleftrightarrow{z} X(z) ; \quad \text{ROC} = R$$

(a) Time Delay

$$x[n-k] \xleftrightarrow{z} z^{-k} X(z)$$

(similar to the Laplace transform of the kth integration of the function)

(where k is a positive integer)

With ROC = R, except for the possible deletion of the origin or addition of the infinity. Since, the multiplication by z^{-k} for $k > 0$, poles will be introduced at $z = 0$, which may cancel corresponding zeros of $X(z)$ at $z = 0$. Consequently, $z = 0$ may be a pole of $z^{-k} X(z)$ while it may not be a pole of $X(z)$. In this case, the ROC for $z^{-k} X(z)$ is the ROC of $X(z)$ but with the origin deleted.

(b) Time Advance

$$x[n+k] \xleftrightarrow{z} z^{k} X(z)$$

(Similar to the Laplace transform of the k^{th} differentiation of the function)

(where k is a positive integer)

With ROC = R, except for the possible deletion of the infinity or addition of the origin. Since, the multiplication by z^k for $k > 0$, zeros will be introduced at $z = 0$, which may cancel corresponding poles of $X(z)$ at $z = 0$. Consequently, $z = 0$ may be a zero of $z^k X(z)$ while it may not be a zero of $X(z)$. In this case, the ROC for $z^k X(z)$ is the ROC of $X(z)$ but with the infinity deleted.

For example : (Assume $k = 1$)

$$x[n-1] \longleftrightarrow z^{-1} X(z); \quad \text{ROC} = R \cap \{0 < |z|\}$$

$$x[n+1] \longleftrightarrow z X(z) \, ; \quad \text{ROC} = R \cap \{|z| < \infty\}$$

From above relationship, z^{-1} is often called the unit-delay operator and z is called the unit-advance operator.

7.5.4. Scaling in the z-Domain (or Complex Translation)

If $\qquad x[n] \overset{z}{\longleftrightarrow} X(z) \, ; \qquad \text{ROC} = R$

then $\qquad z_0^{\,n} x[n] \overset{z}{\longleftrightarrow} X\left(\dfrac{z}{z_0}\right) \, ; \quad \text{ROC} = |z_0| \, R$

In particular, if $X(z)$ has a pole (or zero) at $z = a$, then $X\left(\dfrac{z}{z_0}\right)$ has a pole (or zero) at $z = z_0 \, a$. Here, ROC expands or contracts by the factor $|z_0|$.

7.5.5. Time Expansion

If $\qquad x[n] \overset{z}{\longleftrightarrow} X(z) \, ; \quad \text{ROC} = R$

then $\qquad x_{(k)}[n] \overset{z}{\longleftrightarrow} X(z^k) \, ; \quad \text{ROC} = R^{1/k}$

(where $x_{(k)}[n]$ is introduced and defined in article 6.6.7)

In particular, if $X(z)$ has a pole (or zero) at $z = a$, then $X(z^k)$ has a pole (or zero) at $z = a^{1/k}$. Here, if z is in the ROC of $X(z)$, then the point $z^{1/k}$ is in the ROC of $X(z^k)$.

7.5.6. Time Reversal

If $\qquad x[n] \overset{z}{\longleftrightarrow} X(z) \, ; \qquad \text{ROC} = R$

then $\qquad x[-n] \overset{z}{\longleftrightarrow} X\left(\dfrac{1}{z}\right) \, ; \quad \text{ROC} = \dfrac{1}{R}$

Here, a pole (or zero) of $X(z)$ at $z = a$, then $X\left(\dfrac{1}{z}\right)$ has a pole (or zero) at $z = \dfrac{1}{a}$, *i.e.*, if z_0 is in the ROC for $x[n]$, then $\dfrac{1}{z_0}$ is in the ROC for $x[-n]$. This relationship indicates the inversion of ROC, reflecting the fact that a right-sided discrete-time signal becomes left-sided if time-reversed, and vice-versa.

7.5.7. Conjugation

If $\qquad x[n] \overset{z}{\longleftrightarrow} X(z); \qquad \text{ROC} = R$

then $\qquad x^*[n] \overset{z}{\longleftrightarrow} X^*(z^*); \qquad \text{ROC} = R$

If $X(z)$ has a pole (or zero) at $z = z_0$, it must also have a pole (or zero) at the complex conjugate point $z = z_0^{\,*}$. For a real discrete-time signal $x[n]$, we can easily conclude that

$$X(z) = X^*(z^*).$$

7.5.8. Initial Value Theorem

If the limit exists, then

$$x[0] = \lim_{n \to 0} x[n] = \lim_{z \to \infty} X(z)$$

(For a causal signal, *i.e.*, $x[n] = 0$ for $n < 0$)

Proof : For a causal signal $x[n]$,

$$X(z) = \sum_{n=0}^{\infty} x[n] \,.\, z^{-n} = x[0] + \frac{x[1]}{z} + \frac{x[2]}{z^2} + \ldots$$

As $z \to \infty$, $z^{-n} \to 0$ for $n > 0$ where as for $n = 0$, $z^{-n} = 1$.

Thus, $$\lim_{z \to \infty} X(z) = x[0]$$

7.5.9. Final Value Theorem

$$x[\infty] = \lim_{n \to \infty} x[n] = \lim_{z \to 1} (1 - z^{-1}) X(z)$$

(For a causal signal and the function $X(z)$ has all its poles strictly inside the unit circle except possibly for a first order pole at $z = 1$)

Proof : $$x[n] \overset{z}{\longleftrightarrow} X(z)$$

Using linearity and time-shifting property, we have

$$x[n] - x[n-1] \overset{z}{\longleftrightarrow} (1 - z^{-1}) X(z)$$

or $$\sum_{n=0}^{\infty} \{x[n] - x[n-1]\} z^{-n} = (1 - z^{-1}) X(z)$$

If we now let $z \to 1$, then

$$\lim_{n \to \infty} x[n] = \lim_{z \to 1} (1 - z^{-1}) X(z)$$

7.5.10. Differentiation in the z-Domain (or Multiplication by *n*)

If $$x[n] \overset{z}{\longleftrightarrow} X(z) \quad ; \quad \text{ROC} = R$$

then $$n\, x[n] \overset{z}{\longleftrightarrow} -z \frac{dX(z)}{dz} \quad ; \quad \text{ROC} = R$$

Proof : $$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Differentiating both sides of above equation w.r.t. z, we have

$$\frac{dX(z)}{dz} = \frac{d}{dz}\left\{\sum_{n=-\infty}^{\infty} x[n]\, z^{-n}\right\}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, z^{-n-1} \,.\, (-n) = (-z^{-1}) \sum_{n=-\infty}^{\infty} n x[n]\, z^{-n}$$

or $$-z \frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} n\, x[n]\, z^{-n}$$

7.5.11. Convolution Theorem

If $\quad x[n] \xleftrightarrow{z} X(z)\,;\quad \text{ROC} = R_1$

and $\quad h[n] \xleftrightarrow{z} H(z)\,;\quad \text{ROC} = R_2$

then $\quad x[n] * h[n] \xleftrightarrow{z} X(z)\,.\,H(z)\,;\quad \text{ROC} = R_1 \cap R_2.$

This relationship plays an important role in the analysis and design of discrete-time signals and systems, in analogy with the continuous-time case.

Proof : As we know that,

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[n-k]\,h[k] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$$\mathcal{Z}\{x[n] * h[n]\} = \sum_{n=-\infty}^{\infty} \{x[n] * h[n]\}\,z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left\{\sum_{k=-\infty}^{\infty} x[k]\,h[n-k]\right\} z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} \left\{x[k] \sum_{n=-\infty}^{\infty} h[n-k]\,z^{-n}\right\}$$

Let $\quad n - k = m,\quad$ we have

$$\mathcal{Z}\{x[n] * h[n]\} = \sum_{k=-\infty}^{\infty} \left\{x[k] \sum_{m=-\infty}^{\infty} h[m]\,z^{-m-k}\right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,z^{-k}.\ \sum_{m=-\infty}^{\infty} h[m]\,z^{-m}$$

$$= X(z)\,.\,H(z)$$

EXAMPLE 7.7. Find the z-transform of the unit impulse sequence is defined by the sample values

$$x(nT) = \begin{Bmatrix} 1, & n = 0 \\ 0, & n \neq 0 \end{Bmatrix} \triangleq \delta(nT)$$

Solution: The unit impulse sequence is illustrted in figure 7.9.

Substituting $x(nT)$ into defining equation for the z-transform, we have

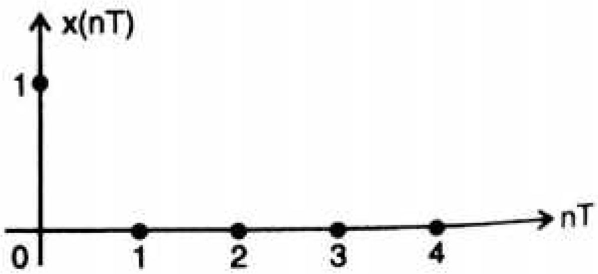

Fig. 7.9.

$$X(z) = \sum_{n=0}^{\infty} x(nT)\cdot z^{-n}$$

$$= 1 + 0\cdot z^{-1} + 0\cdot z^{-2} + \ldots..$$

Thus,

$$X(z) = 1$$

EXAMPLE 7.8. Find the z-transform of the unit step sample sequence is defined by the sample values

$$f(nT) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \triangleq U(nT)$$

Solution: The unit step sample sequence is illustrated in figure 7.10.

Substituting $x(nT)$ into the defining equation, we have

$$X(z) = \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + \ldots$$

Thus $$X(z) = \frac{1}{1-z^{-1}} \quad ; \quad |z| > 1$$

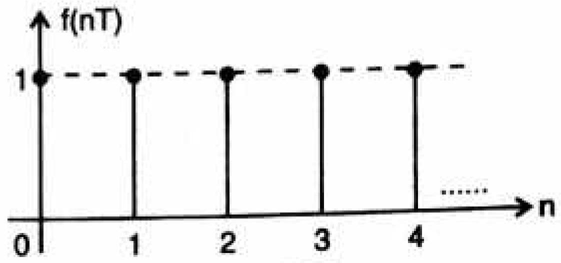

Fig. 7.10.

EXAMPLE 7.9. Find the z-transform of the unit ramp sequence is defind by

$$f(nT) = nT\,U(nT) \triangleq r(nT)$$

Solution: The unit ramp sequence is illustrated in figure 7.11 and has the z-transform

$$F(z) = \sum_{n=0}^{\infty} nT\, z^{-n}$$

$$= 0 + Tz^{-1} + 2\,Tz^{-2} + 3\,Tz^{-3} + \ldots$$

$$= Tz^{-1}(1 + 2\,z^{-1} + 3\,z^{-2} + \ldots)$$

$$\begin{aligned} = T z^{-1} (1 + z^{-1} + z^{-2} + z^{-3} + \ldots \\ + z^{-1} + z^{-2} + z^{-3} + \ldots \\ + z^{-2} + z^{-3} + \ldots \\ + z^{-3} + \ldots) \end{aligned}$$

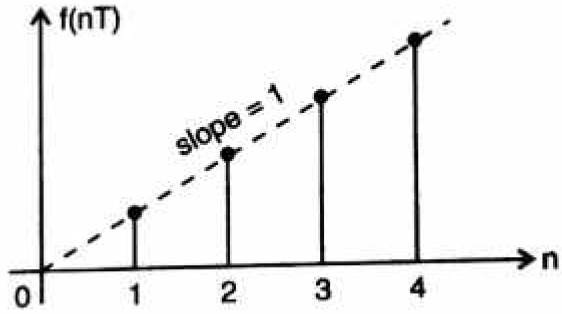

Fig. 7.11.

$$= T z^{-1}\left(\frac{1}{1-z^{-1}} + \frac{z^{-1}}{1-z^{-1}} + \frac{z^{-2}}{1-z^{-1}} + \ldots\right) = \frac{T z^{-1}}{(1-z^{-1})}(1 + z^{-1} + z^{-2} + \ldots)$$

$$= \frac{Tz^{-1}}{(1-z^{-1})^2}; \quad |z| > 1$$

EXAMPLE 7.10. Determine the z-transform of $f(nT) = a^n \operatorname{Sin}\left(\frac{\pi}{2}n\right)$ for $n \geq 0$, where a is a real constant.

Solution:

$$F(z) = \sum_{n=0}^{\infty} a^n \, Sin\left(\frac{\pi}{2}n\right) z^{-n}$$

$$= 0 + a\,z^{-1} + 0 - a^3 z^{-3} + 0 + a^5 z^{-5} + \ldots$$

$$= a\,z^{-1} - a^3 z^{-3} + a^5 z^{-5} - \ldots$$

or $$F(z) = \frac{az^{-1}}{1 + a^2 z^{-2}} \; ; \; |a^2 z^{-2}| < 1 \text{ or } |z| > |a|$$

EXAMPLE 7.11. Determine the z-transform and the ROC of the signal

$$f[n] = [3\,(2^n) - 4\,(3^n)]\,U[n].$$

Since $\mathcal{Z}\left[\left(\frac{3}{4}\right)^n U[n-4]\right] = \mathcal{Z}\left[\left(\frac{3}{4}\right)^4 \cdot \left(\frac{3}{4}\right)^{n-4} U[n-4]\right]$

$$= \left(\frac{3}{4}\right)^4 \cdot \mathcal{Z}\left[\left(\frac{3}{4}\right)^{n-4} \cdot U[n-4]\right]$$

ROC : $|z| > \frac{1}{5}$, $|z| > \frac{3}{4}$ and $|z| > 1$, therefore, ROC is $|z| > 1$.

EXAMPLE 7.15. Find the z-transform of the unit exponential sequence is defined by the sample values

$$f(nT) = e^{-\alpha nT}; \quad \alpha > 0, n \geq 0.$$

Solution: The unit exponential sequence is illustrated in figure 7.12 and has the z-transform

$$F(z) = \sum_{n=0}^{\infty} e^{-\alpha nT} \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(e^{-\alpha T} z^{-1}\right)^n$$

Thus $F(z) = \dfrac{1}{1-e^{-\alpha T} z^{-1}}$; $|z| > e^{-\alpha T}$

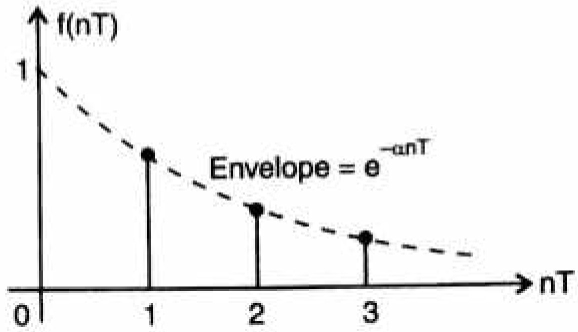

Fig. 7.12.

EXAMPLE 7.16. Find the z-transform of the discrete function

$$f(nT) = \text{Sin } \omega nT.\ U(nT)$$

or $$f(nT) = \text{Sin } \omega nT, \quad n \geq 0$$

Solution: $F(z) = \sum_{n=0}^{\infty} \sin \omega nT . z^{-n}$

$$= \sum_{n=0}^{\infty} \frac{1}{j2}\left[e^{j\omega nT} - e^{-j\omega nT}\right] . z^{-n}$$

$$= \frac{1}{j2}\left[\sum_{n=0}^{\infty} (e^{j\omega T} \cdot z^{-1})^n - \sum_{n=0}^{\infty} (e^{-j\omega T} z^{-1})^n\right]$$

$F(z) =$; $|z| > e^{j\omega T}$ and $|z| > e^{-j\omega T}$;

$$F(z) = \frac{1}{j2}\left[\frac{1}{1-e^{j\omega T} z^{-1}} - \frac{1}{1-e^{-j\omega T} z^{-1}}\right]; \ |z| > 1$$

$$F(z) = \frac{(Sin\ \omega T) z^{-1}}{1 - 2z^{-1} Cos\ \omega T + z^{-2}}; \ |z| > 1$$

EXAMPLE 7.17. Find the z-transform of $f(nT) = e^{-\alpha nT} \sin \omega nT.\ U(nT)$

Solution: $F(z) = \sum_{n=0}^{\infty} e^{-\alpha nT} \sin \omega nT . z^{-n}$

$$= \frac{1}{j2}\left[\sum_{n=0}^{\infty} (e^{-\alpha T} e^{j\omega T} z^{-1})^n - \sum_{n=0}^{\infty} (e^{-\alpha T} e^{-j\omega T} z^{-1})^n\right] \quad |z| > e^{-(\alpha - j\omega)T} \text{ and } |z| > e^{-(\alpha + j\omega)T}$$

$$F(z) = \frac{1}{j2}\left[\frac{1}{1-e^{-\alpha T}e^{j\omega T}z^{-1}} - \frac{1}{1-e^{-\alpha T}e^{-j\omega T}z^{-1}}\right]$$

$$= \frac{e^{-\alpha T}(\sin \omega T)z^{-1}}{1-2e^{-\alpha T}(\cos \omega T)z^{-1}+e^{-2\alpha T}z^{-2}}\ ;\ |z| > e^{-\alpha T}$$

⇨ 7.6. INVERSE Z-TRANSFORM

Just as in the Laplace transform, one of the major objectives of the z-transform is that algebraic manipulations can be made first in the z-domain, and then the final time response is determined by the inverse z-transform, *i.e.*, the procedure for transforming from the z-domain to the time domain is called the inverse z-transform. There are, however, a number of alternative procedures for obtaining a discrete function from its z-transform. As with Laplace transforms, one perticularly useful for rational z-transforms consists of expanding the algebraic expression into a partial fraction expansion and recognizing the sequences associated with the individual terms using z-transform pairs of Table 7.1. In the following examples, we illustrate the procedure.

EXAMPLE 7.18. Find the inverse z-transform for given the z-transform function

$$F(z) = \frac{(1-e^{-\alpha T})z^{-1}}{(1-z^{-1})(1-e^{-\alpha T}z^{-1})}\ ;\ |z| > e^{-\alpha T}$$

Solution: Using partial fraction expansion, we have

$$F(z) = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-\alpha T}z^{-1}}$$

The corresponding inverse z-transform of $F(z)$ is found to be

$$f[n] = U[n] - e^{-\alpha nT}\,U[n]$$

or

$$f[n] = (1 - e^{-\alpha nT})\,U[n]$$

EXAMPLE 7.19. Determine the inverse z-transform of the proper function

$$F(z) = \frac{1}{1-1\cdot 5z^{-1}+0.5z^{-2}}\ ;\ |z| > 1$$

Solution:

$$F(z) = \frac{1}{(1-z^{-1})(1-0.5z^{-1})}$$

Using partial fraction expansion, we have

$$F(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}}$$

Therefore,

$$f[n] = [2-(0.5)^n]\,U[n]$$

EXAMPLE 7.20. Determine the inverse z-transform of the function

$$F(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$$

Solution: The poles of $F(z)$ are complex conjugate

$$p_1 = \frac{1}{2}(1+j)$$

and

$$p_2 = \frac{1}{2}(1-j)$$

$$F(z) = \frac{A}{1-p_1 z^{-1}} + \frac{B}{1-p_2 z^{-1}}$$

Where $$A = \frac{1}{2}(1-j3)$$

and $$B = \frac{1}{2}(1+j3)$$

Therefore, $$f[n] = \left[\frac{1}{2}(1-j3)\left\{\frac{1}{2}(1+j)\right\}^n + \frac{1}{2}(1+j3)\left\{\frac{1}{2}(1-j)\right\}^n\right]U[n]$$

EXAMPLE 7.21. Find $f(n)$, if $F(z) = 4z^2 + 3z + 7 + 2z^{-1}$, $0 < |z| < \infty$.

Solution: If we simply use the transform pair

$$\delta[n+k] \overset{z}{\longleftrightarrow} z^k$$

Then we have

$$f[n] = 4\delta[n+2] + 3\delta[n+1] + 7\delta[n] + 2\delta[n-1]$$

EXAMPLE 7.22. Find $f(n)$, if

$$F(z) = \frac{3-\frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} \text{ with ROC is}$$

(a) $|z| > \frac{1}{3}$, (b) $\frac{1}{4} < |z| < \frac{1}{3}$ and (c) $|z| < \frac{1}{4}$.

Solution: Using partial fraction expansion

$$F(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{3}z^{-1}}$$

Thus, $f(n)$ is the sum of two terms, one with z-transform $\frac{1}{1-\frac{1}{4}z^{-1}}$ and the other with z-transform $\frac{2}{1-\frac{1}{3}z^{-1}}$. In order to determine the inverse z-transform of each of these individual terms, we must specify the ROC associated with each.

Thus,

$$f[n] = f_1[n] + f_2[n]$$

where $$f_1[n] \overset{z}{\leftrightarrow} \frac{1}{1-\frac{1}{4}z^{-1}}; \quad |z| > \frac{1}{4}$$

$$f_2[n] \overset{z}{\leftrightarrow} \frac{2}{1-\frac{1}{3}z^{-1}}; \quad |z| > \frac{1}{3}$$

And thus,

(a) $$f[n] = \left(\frac{1}{4}\right)^n U[n] + 2\left(\frac{1}{3}\right)^n U[n]$$

(b) $f[n] = \left(\frac{1}{4}\right)^n U[n] - 2\left(\frac{1}{3}\right)^n U[-n-1]$

(c) $f[n] = -\left(\frac{1}{4}\right)^n U[-n-1] - 2\left(\frac{1}{3}\right)^n U[-n-1]$

EXAMPLE 7.23. Determine the inverse z-transform of the function

$$F(z) = \frac{1}{z^2 - 1.2z + 0.2}$$

Solution:

$$F(z) = \frac{1}{(z-1)(z-0.2)}$$

In this case, the first step in performing the inverse z-transform is to divide both sides by z. This yields

$$\frac{F(z)}{z} = \frac{1}{z(z-1)(z-0.2)}$$

Using partial fraction expansion,

$$\frac{F(z)}{z} = \frac{5}{z} + \frac{1.25}{z-1} - \frac{6.25}{z-0.2}$$

Thus

$$F(z) = 5 + 1.25\,\frac{z}{z-1} - 6.25\,\frac{z}{z-0.2}$$

or

$$F(z) = 5 + 1.25\,\frac{1}{1-z^{-1}} - 6.25\,\frac{1}{1-0.2z^{-1}}$$

We can now take inverse z-transform. This yields

$$f[n] = 5\,\delta[n] + 1.25\,U[n] - 6.25\,(0.2)^n\,U[n]$$

Alternatively : Using partial fraction expansion,

$$F(z) = \frac{1.25}{z-1} - \frac{1.25}{z-0.2}$$

$$= z^{-1}\left[\frac{(1.25)z}{z-1} - \frac{(1.25)z}{z-0.2}\right] = z^{-1}\left[\frac{1.25}{1-z^{-1}} - \frac{1.25}{1-0.2z^{-1}}\right]$$

Taking inverse z-transform, we obtain

$$f[n] = 1.25\left[1-(0.2)^{n-1}\right] U[n-1]$$

Note :

In example 7.23, for the same $F(z)$ we got the two different expressions for $f[n]$ by two different procedures. We can easily verify that these expressions are same.

$$\begin{aligned}
f[n] &= 5\delta[n] + 1.25\,U[n] - 6.25\,(0.2)^n\,U[n] \\
&= 5\delta[n] + 1.25\,\{\delta[n] + U[n-1]\} - 6.25\,(0.2)^n\,\{\delta[n] + U[n-1]\} \\
&= 5\delta[n] + 1.25\,\delta[n] + 1.25\,U[n-1] - 6.25\,(0.2)^n\,\delta[n] \\
&\qquad - 6.25(0.2)\,(0.2)^{n-1}\,U[n-1] \\
&= 6.25\delta[n] + 1.25\,U[n-1] - 6.25\delta[n] - 1.25\,(0.2)^{n-1}\,U[n-1] \\
&= 1.25\,U[n-1] - 1.25\,(0.2)^{n-1}\,U[n-1] \\
&= 1.25\,[1-(0.2)^{n-1}]\,U[n-1].
\end{aligned}$$

⇨ 7.7. THE SYSTEM FUNCTION OF A LINEAR TIME INVARIANT SYSTEM

As we know that the output $y[n]$ of a linear time-invariant system to an input $x[n]$ can be obtained by computing the convolution sum of $x[n]$ with the unit sample response $h[n]$ of the system. To express this relationship in the z-domain as

$$Y(z) = H(z) \cdot X(z)$$

where $Y(z)$ is the z-transform of the output $y[n]$, $X(z)$ is the z-transform of the input $x[n]$ and $H(z)$ is the z-transform of the unit sample response $h[n]$.

If we know $h[n]$ and $x[n]$, we can determine their corresponding z-transforms $H(z)$ and $X(z)$, multiply them to obtain $Y(z)$, and therefore determine $y[n]$ by evaluating the inverse z-transform of $Y(z)$. Alternatively, if we know $x[n]$ and we observe the output $y[n]$ of the system, we can determine the unit sample response by first solving for $H(z)$ from the relation

$$H(z) = \frac{Y(z)}{X(z)}$$

and then evaluating the inverse z-transform of $H(z)$. The transform $H(z)$ is called the **system function** or **transfer function** or **pulse transfer function**.

7.7.1. Causality

A discrete-time system is causal if it has an impulse response $h[n]$, that is zero for $n < 0$, *i.e.*, $h[n]$ is a right-sided signal. The corresponding requirement on $H(z)$ is that the ROC of $H(z)$ is the exterior of a circle containing all of the poles of $H(z)$ in the z-plane.

And if the system is anti-causal, that is zero for $n \geq 0$, *i.e.*, $h[n]$ is a left-sided signal. The corresponding ROC of $H(z)$ is the interior of a circle containing no poles of $H(z)$ in the z-plane.

7.7.2. Stability

A discrete-time system is BIBO stable if its impulse response being absolutely summable, *i.e.*,

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

and consequently, the corresponding requirement on $H(z)$ is that the ROC of $H(z)$ includes the unit circle, *i.e.*, $|z| = 1$.

EXAMPLE 7.24. Determine the system function and the unit sample response of the system described by the difference equation

$$y[n] = \frac{1}{2} y[n-1] + 2x[n]$$

Solution: By computing the z-transform of the difference equation, we obtain

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2\, X(z)$$

Hence, the system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - \frac{1}{2} z^{-1}}$$

Taking inverse z-transform, we obtain

$$h[n] = 2\left(\frac{1}{2}\right)^n U[n]$$

This is the unit sample response of the system.

EXAMPLE 7.25. Determine the response of the system

$$y[n] = \frac{5}{6}y[n-1]-\frac{1}{6}y[n-2]+x[n]$$

to the input signal $x[n] = \delta[n]-\frac{1}{3}\delta[n-1]$

Solution: Taking the z-transform, we have

$$Y(z) = \frac{5}{6}z^{-1}Y(z)-\frac{1}{6}z^{-2}Y(z)+X(z)$$

and $$X(z) = 1-\frac{1}{3}z^{-1}$$

The system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}} = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

$$Y(z) = H(z)\cdot X(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

and hence the response of the system is

$$y[n] = \left(\frac{1}{2}\right)^n U[n]$$

EXAMPLE 7.26. Find the z-transform of the following discrete functions.

(*i*) $r(nT)$ (*ii*) sin ωnT. $U(nT)$ *(I.P. Univ., 2001)*

Solution: (*i*) Same as example 7.8.

(*ii*) Same as example 7.15.

EXAMPLE 7.27. Find inverse z-transform of the following

(*i*) $$F(z) = \frac{1}{2(z+0.5)(z-1)}$$

(*ii*) $$F(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$ *(I.P. Univ., 2001)*

Solution: (*i*) $$F(z) = \frac{1}{2(z+0.5)(z-1)}$$

or $$\frac{F(z)}{z} = \frac{1}{2z(z+0.5)(z-1)}$$

Using partial fraction expansion,

$$\frac{F(z)}{z} = -\frac{1}{z}+\frac{2}{3}\cdot\frac{1}{z+0.5}+\frac{1}{3}\cdot\frac{1}{z-1}$$

Thus, $$F(z) = -1+\frac{2}{3}\frac{z}{z+0.5}+\frac{1}{3}\frac{z}{z-1} = -1+\frac{\frac{2}{3}}{1+0.5z^{-1}}+\frac{\frac{1}{3}}{1-z^{-1}}$$

Taking inverse z-transform, we have

$$f[n] = -\delta[n]+\frac{2}{3}(-0.5)^n U[n]+\frac{1}{3}U[n]$$

(ii) $$F(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

Using partial fraction expansion,

$$F(z) = \frac{2}{1-\frac{1}{2}z^{-1}}-\frac{1}{1-\frac{1}{4}z^{-1}}$$

Taking inverse z-transform, we have

$$f[n] = 2\left(\frac{1}{2}\right)^n U[n]-\left(\frac{1}{4}\right)^n U[n] = \left[2\left(\frac{1}{4}\right)^n\cdot 2^n-\left(\frac{1}{4}\right)^n\right]U[n]$$

or $$f[n] = \left(\frac{1}{4}\right)^n\left[2^{n+1}-1\right]U[n]$$

EXAMPLE 7.28. Find the inverse z-transform of

(i) $F(z) = \frac{z}{z^2-z+1}$ **and (ii)** $F(z) = \frac{z-1}{(z-\alpha)^2}$. *(I.P.Univ., 2000)*

Solution: *(i)* $$F(z) = \frac{z}{z^2-z+1} = \frac{z}{\left(z-\frac{1}{2}\right)^2-\left(j\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{z}{\left[z-\left(\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\right]\left[z-\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\right]}$$

Using partial fraction expansion,

$$F(z) = \frac{(\sqrt{3}+j)/2\sqrt{3}}{z-\left(\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}+\frac{(\sqrt{3}-j)/2\sqrt{3}}{z-\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)}$$

or $$F(z) = \left[\frac{\sqrt{3}+j}{2\sqrt{3}}\cdot\frac{z^{-1}}{1-\left(\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)z^{-1}}+\frac{\sqrt{3}-j}{2\sqrt{3}}\cdot\frac{z^{-1}}{1-\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)z^{-1}}\right]$$

Taking inverse z-transform, we have

$$f[n] = \frac{\sqrt{3}+j}{2\sqrt{3}}\cdot\left(\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)^{n-1}+\frac{\sqrt{3}-j}{2\sqrt{3}}\left(\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)^{n-1}$$

(ii) $$F(z) = \frac{z-1}{(z-\alpha)^2} = \frac{z}{(z-\alpha)^2} - \frac{1}{(z-\alpha)^2} = \frac{z^{-1}}{(1-\alpha z^{-1})^2} - \frac{z^{-2}}{(1-\alpha z^{-1})^2}$$

$$F(z) = \frac{1}{\alpha}\cdot\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} - \frac{1}{\alpha} z^{-1}\left(\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}\right)$$

Taking inverse z-transform, we have

$$f[n] = \frac{1}{\alpha} n\alpha^n - \frac{1}{\alpha}(n-1)\alpha^{n-1} = n\alpha^{n-1} - (n-1)\alpha^{n-2}$$

or $$\mathbf{f[n] = \alpha^{n-2}[n(\alpha-1)+1]}$$

EXAMPLE 7.29. Find a linear constant coefficient difference equation relating the input and output of the impulse response given by

$$h[n] = \left(\frac{1}{2}\right)^n U[n] + \frac{1}{2}\left(\frac{1}{4}\right)^n U[n].$$

Solution: Taking z-transform, we have

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{2}\cdot\frac{1}{1-\frac{1}{4}z^{-1}} = \frac{3-z^{-1}}{2\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

Since, we know that

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3-z^{-1}}{2-\frac{3}{2}z^{-1}+\frac{1}{4}z^{-2}}$$

Therefore,

$$\left(2-\frac{3}{2}z^{-1}+\frac{1}{4}z^{-2}\right)Y(z) = (3-z^{-1})X(z)$$

or $$2Y(z) - \frac{3}{2}z^{-1}Y(z) + \frac{1}{4}z^{-2}Y(z) = 3X(z) - z^{-1}X(z)$$

Taking inverse z-transform, we obtain

$$\mathbf{2y[n] - \frac{3}{2}y[n-1] + \frac{1}{4}y[n-1] = 3x[n] - x[n-1]}$$

EXAMPLE 7.30. Find the response $y[n]$ of an LTI discrete system described by equation

$$\mathbf{y[n+2] + 0.6y[n+1] - 0.16y[n] = 5x[n+2]}$$

if input $$\mathbf{x[n] = 4^{-n}U[n]}$$

Solution: $$x[n] = 4^{-n}U[n] = \left(\frac{1}{4}\right)^n U[n]$$

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-1}} \quad ...(1)$$

and $$y[n+2] + 0.6y[n+1] - 0.16y[n] = 5x[n+2]$$

Taking z-transform, we get

$$z^2Y(z) + 0.6zY(z) - 0.16Y(z) = 5z^2X(z) \quad ...(2)$$

Putting the values of $X(z)$ from equation (1), equation (2) becomes

$$(z^2 + 0.6z - 0.16)\, Y(z) = \frac{5z^2}{1-\frac{1}{4}z^{-1}}$$

Therefore, $$Y(z) = \frac{5z^2}{\left(1-\frac{1}{4}z^{-1}\right)(z^2+0.6z-0.16)}$$

$$Y(z) = \frac{5}{\left(1-\frac{1}{4}z^{-1}\right)(1+0.6z^{-1}-0.16z^{-2})} = \frac{5}{\left(1-\frac{1}{4}z^{-1}\right)(1+0.8z^{-1})(1-0.2z^{-1})}$$

or $$Y(z) = \frac{5}{\left(1-\frac{1}{4}z^{-1}\right)\left(1+\frac{4}{5}z^{-1}\right)\left(1-\frac{1}{5}z^{-1}\right)}$$

Using partial fraction expansion, we obtain

$$Y(z) = \frac{\left(\frac{125}{21}\right)}{\left(1-\frac{1}{4}z^{-1}\right)} + \frac{\left(\frac{64}{21}\right)}{\left(1+\frac{4}{5}z^{-1}\right)} - \frac{4}{\left(1-\frac{1}{5}z^{-1}\right)}$$

Taking inverse z-transform, we get

$$y[n] = \left[\frac{125}{21}\left(\frac{1}{4}\right)^n + \frac{64}{21}\left(-\frac{4}{5}\right)^n - 4\left(\frac{1}{5}\right)^n\right] U[n]$$

EXAMPLE 7.31. Find the pole zero plot for the signal

$$x[n] = (2)^n\, U[n]$$

Solution: $$X(z) = \frac{1}{1-2z^{-1}}; \quad |z| > 2$$

$$= \frac{z}{z-2}; \quad |z| > 2$$

pole at $z = 2$

zero at $z = 0$

The pole-zero diagram is shown in figure 7.13.

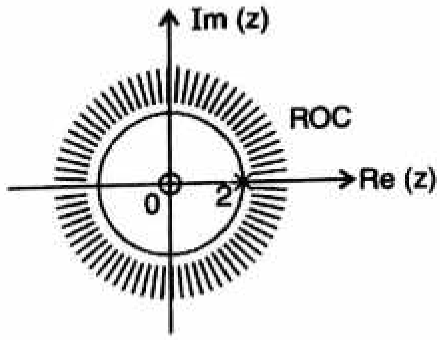

Fig. 7.13. Pole-zero diagram of $x[n] = (2)^n U(n)$

EXAMPLE 7.32. Using (*i*) Long division, (*ii*) Partial fraction expansion, determine the inverse z-transform of

$$X(z) = \frac{1}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

Also verify the results in each case for atleast three values of n.

(*a*) when ROC : $|z| > 1$; (*b*) when ROC : $|z| < \frac{1}{2}$.

Solution: (*a*) Since ROC : $|z| > 1$, therefore $x[n]$ is a causal signal. Hence,

$$1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \ldots..$$

$$
\begin{array}{r|l}
1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} & 1 \\
& 1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} \\ \hline
& \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} \\
& \frac{3}{2}z^{-1} - \frac{9}{4}z^{-2} + \frac{3}{4}z^{-3} \\ \hline
& \frac{7}{4}z^{-2} - \frac{3}{4}z^{-3} \\
& \frac{7}{4}z^{-2} - \frac{21}{8}z^{-3} + \frac{7}{8}z^{-4} \\ \hline
& \frac{15}{8}z^{-3} - \frac{7}{8}z^{-4}
\end{array}
$$

Therefore, we have

$$X(z) = 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \ldots$$

Taking inverse z-transform, we get

$$x[n] = \left\{1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots\ldots\right\}$$

And using partial fraction expansion,

$$X(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}; \quad |z| > 1$$

Taking inverse z-transform, we get

$$x[n] = \left[2 - \left(\frac{1}{2}\right)^n\right] U[n]$$

for $n = 0, 1, 2, 3, \ldots\ldots$ gives

$$x[n] = \left\{1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots\ldots\right\} \quad \textbf{Verified.}$$

Note: A discrete-time LTI system which has a transfer function $H(z)$ will be causal if and only if

(*i*) the ROC is the exterior of a circle (including infinity) but outside the outermost pole and

(*ii*) the degree of the numerator of $H(z)$ must be smaller or equal to than the degree of denominator.

Example :

$$H(z) = \frac{z^3 - 3z^2 + 2z}{z^2 + \frac{1}{2}z - \frac{1}{4}} \rightarrow \text{Anti-causal}$$

$$H(z) = \frac{4z^2 - 5z}{2z^2 - 5z + 2} \rightarrow \text{Causal}$$

(*b*) Since $ROC : |z| < \frac{1}{2}$, therefore $x[n]$ is non-causal. Hence,

$$\begin{array}{r|l} & 2z^2 + 6z^3 + 14z^4 + 30z^5 + \ldots \\ \hline \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & 1 \\ & 1 - 3z + 2z^2 \\ \hline & 3z - 2z^2 \\ & 3z - 9z^2 + 6z^3 \\ \hline & 7z^2 - 6z^3 \\ & 7z^2 - 21z^3 + 14z^4 \\ \hline & 15z^3 - 14z^4 \end{array}$$

Therefore, we have

$$X(z) = 2z^2 + 6z^3 + 14z^4 + 30z^5 + \ldots.$$

or

$$X(z) = \ldots\ldots.. + 30z^5 + 14z^4 + 6z^3 + 2z^2$$

Taking inverse z-transform, we get

$$x[n] = \{\ldots\ldots..\ 30, 14, 6, 2, 0\}$$

And using partial fraction expansion.

$$X(z) = \frac{2}{1 - z^{-1}} - \frac{1}{1 - \frac{1}{2}z^{-1}}; \quad |z| < \frac{1}{2}$$

Taking inverse z-transform, we get

$$x[n] = \left[-2 - (-1)\left(\frac{1}{2}\right)^n\right] U[-n-1]$$

$$\left[\left(\frac{1}{2}\right)^n - 2\right] U[-n-1]$$

For $n = \ldots\ldots. -5, -4, -3, -2, -1.$

$$x[n] = \{\ldots\ldots...\ 30, 14, 6, 2, 0\}$$ **Verified.**

EXAMPLE 7.33. Find the discrete-time signal $x[n]$ whose z-transform is given as

$$X(z) = \log(1 + az^{-1}); \ |z| > |a|$$

Solution:

$$\frac{dX(z)}{dz} = \frac{1}{1 + az^{-1}}(-az^{-2})$$

Hence

$$-z\frac{dX(z)}{dz} = az^{-1}\left[\frac{1}{1 + az^{-1}}\right]; \quad |z| > |a|$$

$$n\,x[n] = a(-a)^{n-1}U[n-1]$$

or

$$x[n] = (-1)^{n-1} \cdot \frac{a^n}{n} \cdot U[n-1]$$

EXAMPLE 7.34. Find the sample response $h[n]$ of an LTI system described by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2\,x[n]$$

Also calculate the response $y[n]$ to the input

$$x[n] = \left(\frac{1}{4}\right)^n U[n].$$

Solution: Taking z-transform of the given difference equation, we have

$$Y(z) - \frac{3}{4} z^{-1} Y(z) + \frac{1}{8} z^{-2} Y(z) = 2 X(z)$$

or $$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}}$$

$$= \frac{2}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 - \frac{1}{4} z^{-1}\right)}$$

Using partial fraction expansion,

$$H(z) = \frac{4}{1 - \frac{1}{2} z^{-1}} - \frac{2}{1 - \frac{1}{4} z^{-1}}$$

Taking inverse z-transform by inspection, we have

$$h[n] = \left[4\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{4}\right)^n\right] U[n]$$

We know that

$$Y(z) = H(z) \cdot X(z) = \frac{2}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 - \frac{1}{4} z^{-1}\right)^2}$$

$$\left[\text{Since } x[n] = \left(\frac{1}{4}\right)^n U[n] \text{ or } X(z) = \frac{1}{1 - \frac{1}{4} z^{-1}}\right]$$

Using partial fraction expansion,

$$Y(z) = \frac{8}{1 - \frac{1}{2} z^{-1}} - \frac{4}{1 - \frac{1}{4} z^{-1}} - \frac{2}{\left(1 - \frac{1}{4} z^{-1}\right)^2}$$

Taking inverse z-transform, we get

$$y[n] = \left[8\left(\frac{1}{2}\right)^n - 4\left(\frac{1}{n}\right)^n\right] U[n] - 2(n+1) \left(\frac{1}{4}\right)^n U[n+1]$$

$$y[n] = \left[8\left(\frac{1}{2}\right)^n - 4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n\right] U[n]$$

(Since last term of right hand side is zero at $n = -1$)

or $$y[n] = \left[8\left(\frac{1}{2}\right)^n - 6\left(\frac{1}{4}\right)^n - 2n\left(\frac{1}{4}\right)^n\right] U[n]$$

EXAMPLE 7.35. Determine the initial and final values of the discrete time signal $x[n]$, whose z-transform is given by $X(z) = 3 + 5z^{-1} + 7z^{-2}$.

Solution: We know that,

The initial value $x[0] = \lim_{n\to 0} x[n] = \lim_{z\to\infty} X(z)$

$$= \lim_{z\to\infty} (3 + 5z^{-1} + 7z^{-2}) = 3$$

and the final value $x[\infty] = \lim_{n\to\infty} x[n] = \lim_{z\to 1} (1 - z^{-1})\, X(z)$

$$= \lim_{z\to 1} (1 - z^{-1})\,(3 + 5z^{-1} + 7z^{-2})$$

$$= \lim_{z\to 1} (3 + 2z^{-1} + 2z^{-2} - 7z^{-3}) = 0$$

Table 7.1. Some Common z-Transform Pairs

Transform Pair Number	*Discrete-time Function $f(nT)$ for $n \ge 0$*	*z-transform of $f(nT)$*	*ROC*
1	$\delta(nT) = \begin{Bmatrix} 1, & n = 0 \\ 0, & n \ne 0 \end{Bmatrix}$	1	All Z
2.	$U(nT) = \begin{Bmatrix} 1, & n \ge 0 \\ 0, & n < 0 \end{Bmatrix}$	$\dfrac{1}{1 - z^{-1}}$	$\lvert z \rvert > 1$
3.	$r(nT) = nT.$	$\dfrac{Tz^{-1}}{(1 - z^{-1})^2}$	$\lvert z \rvert > 1$
4.	$e^{-\alpha nT} = \left(e^{-\alpha T}\right)^n = K^n$	$\dfrac{1}{1 - e^{-\alpha T} z^{-1}} = \dfrac{1}{1 - Kz^{-1}}$	$\lvert z \rvert > K$
5.	$nT\, e^{-\alpha nT}$	$\dfrac{Te^{-\alpha T} z^{-1}}{\left(1 - e^{-\alpha T} z^{-1}\right)^2}$	$\lvert z \rvert > e^{-\alpha T}$
6.	$\sin \omega nT$	$\dfrac{(\sin \omega T) z^{-1}}{1 - 2(\cos \omega T)\, z^{-1} + z^{-2}}$	$\lvert z \rvert > 1$
7.	$\cos \omega nT$	$\dfrac{1 - (\cos \omega T)\, z^{-1}}{1 - 2(\cos \omega T)\, z^{-1} + z^{-2}}$	$\lvert z \rvert > 1$
8.	$e^{-\alpha nT} \sin \omega nT$	$\dfrac{e^{-\alpha T} (\sin \omega T)\, z^{-1}}{1 - 2e^{-\alpha T} (\cos \omega T)\, z^{-1} + e^{-2\alpha T} z^{-2}}$	$\lvert z \rvert > e^{-\alpha T}$
9.	$e^{-\alpha nT} \cos \omega nT$	$\dfrac{1 - e^{-\alpha T} (\cos \omega T)\, z^{-1}}{1 - 2e^{-\alpha T} (\cos \omega T)\, z^{-1} + e^{-2\alpha T} z^{-2}}$	$\lvert z \rvert > e^{-\alpha T}$

7.7.3. Interconnections of the Systems

If two LTI discrete-time systems with impulse responses $h_1[n]$ and $h_2[n]$ are connected in cascade, the overall impulse response $h[n]$ of the interconnected system is given by

$$h[n] = h_1[n] * h_2[n]$$

and the corresponding system function $H(z)$ is given by

$$H(z) = H_1(z) \cdot H_2(z)$$

where
$$h_1[n] \overset{z}{\longleftrightarrow} H_1(z)\,; \quad \text{ROC} = R_1$$

$$h_2[n] \overset{z}{\longleftrightarrow} H_2(z)\,; \quad \text{ROC} = R_2$$

$$h[n] \overset{z}{\longleftrightarrow} H(z)\,; \quad \text{ROC} = R_1 \cap R_2$$

Similarly, if two LTI discrete-time systems with impulse responses $h_1[n]$ and $h_2[n]$ are connected in parallel, the overall impulse response $h[n]$ is given by

$$h[n] = h_1[n] + h_2[n]$$

and
$$H[z] = H_1(z) + H_2(z)\,; \quad \text{ROC} = R_1 \cap R_2$$

where R_1 and R_2 are the ROCs of the functions $H_1(z)$ and $H_2(z)$, respectively.

7.7.4. Systems Block-Diagrams

As in Article 7.3, we can represent the LTI discrete-time systems described by difference equations using block-diagrams involving three basic operations; addition, multiplication by a coefficient, and a unit delay. We discussed such a block diagram for a first-order difference equation. Here, we consider system functions in constructing block-diagram representation as illustrated in following examples:

EXAMPLE 7.36. Draw the block-diagram representation for the causal LTI system with system function.

$$H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

Solution: By the definition of system function, *i.e.*,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

Now, taking inverse z-transform, we can easily see that this system can also be described by the difference equation

$$y[n] - \frac{1}{2}\, y[n-1] = x[n]$$

From Article 7.3, we constructed a block-diagram representation of this system as shown in figure 7.14. Here, z^{-1} is the system function of a unit delay.

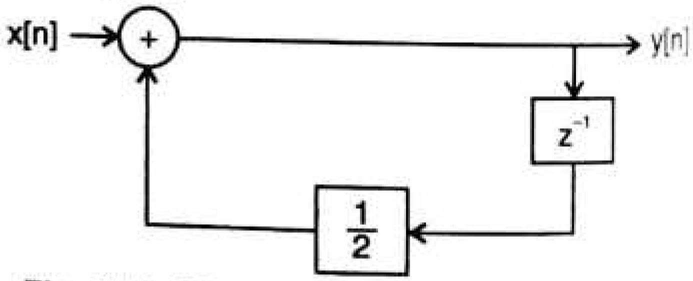

Fig. 7.14. Block-diagram representation for example 7.36

EXAMPLE 7.37. Construct a block-diagram of a LTI discrete-time system with system function

$$H(z) = \frac{1 - \frac{1}{4} z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

Note : If the system is both causal and stable, then all of the poles of its system function $H(z)$ must lie inside the unit circle of the z-plane.

Solution: Let $H(z) = H_1(z) \cdot H_2(z)$

where $$H_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

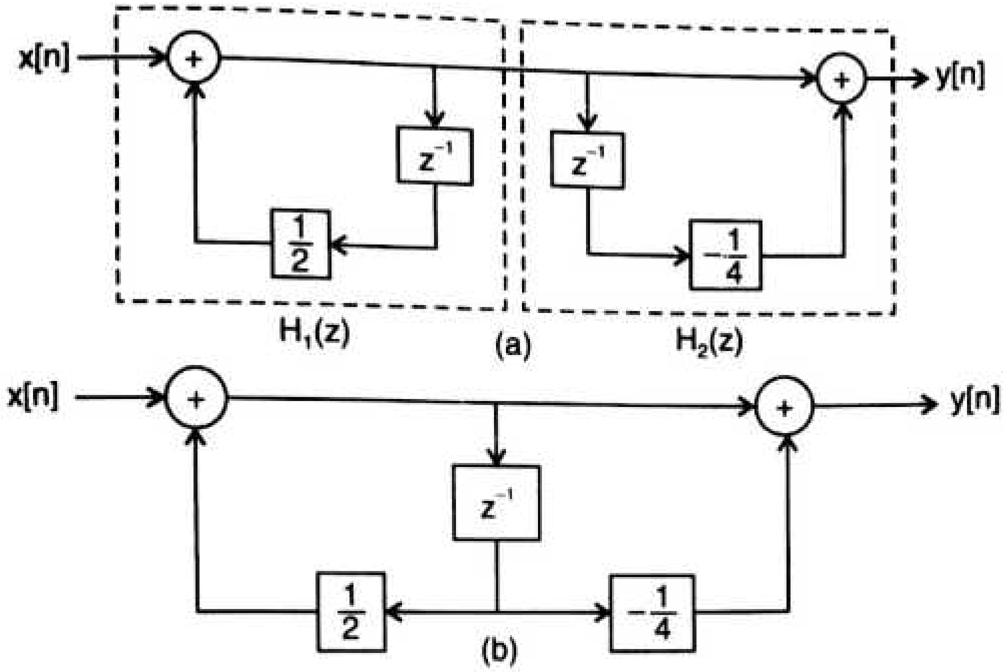

Fig. 7.15. (a) Block-diagram representation for example 7.37
(b) equivalent block-diagram representation using only one unit delay element.

and $$H_2(z) = 1 - \frac{1}{4}z^{-1}$$

We can say this system is the cascade interconnection of two sub-systems with system functions $H_1(z)$ and $H_2(z)$.

EXAMPLE 7.38. Draw the block-diagram representation for a second-order system function

$$H(z) = \frac{1}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

Solution: Using the same concepts as in previous examples, we construct the block-diagram representation for this system as shown in figure 7.16 (*a*). This is commonly referred to as a direct-form.

We can rewrite the given system function by factorizing the dominator as

$$H(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

which suggest the cascade-form representation as shown in figure 7.16 (*b*).

Also, by performing the partial fraction expansion of the system function $H(z)$, we have

$$H(z) = \frac{2}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-\frac{1}{4}z^{-1}}$$

which leads to the parallel-form representation as shown in figure 7.16 (*c*).

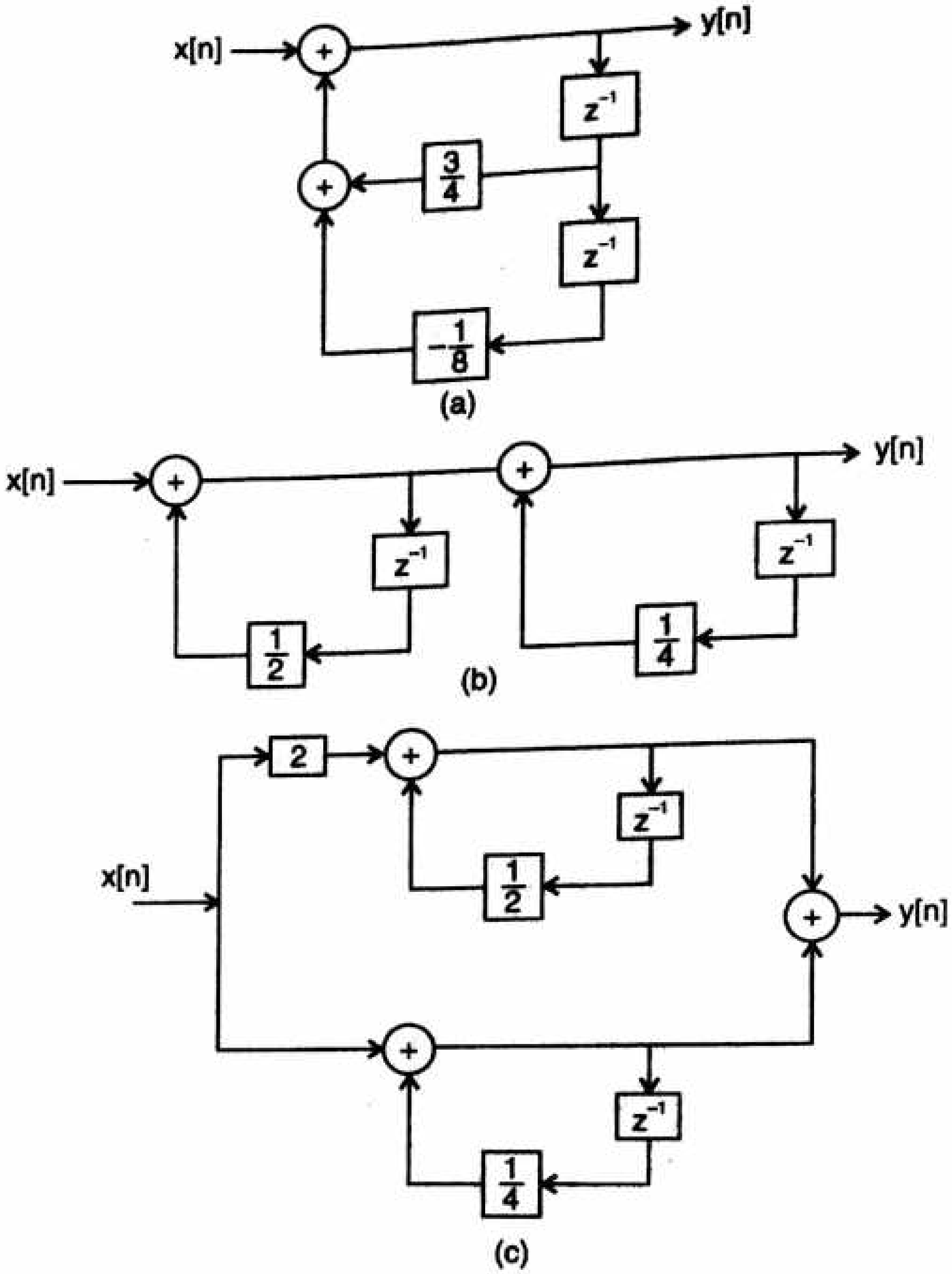

Fig. 7.16. Block-diagram representation for example 7.38 (a) direct-form, (b) cascade-form, (c) parallel-form

EXAMPLE 7.39. Draw the direct-form block-diagram representation for a system function

$$H(z) = \frac{1 - \frac{1}{2}z^{-1} - \frac{1}{4}z^{-2}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

Solution: Rewrite the system function as

$$H(z) = \left(\frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} \right) \left(1 - \frac{1}{2}z^{-1} - \frac{1}{4}z^{-2} \right)$$

leads to as the cascade of the system of previous example as shown in figure 7.16 (a) and the system with system function $\left(1-\frac{1}{2}z^{-1}-\frac{1}{4}z^{-2}\right)$. The result is the direct-form block-diagram shown in figure 7.17. The coefficients in the direct-form representation can be found by inspection from the coefficients in the given system function.

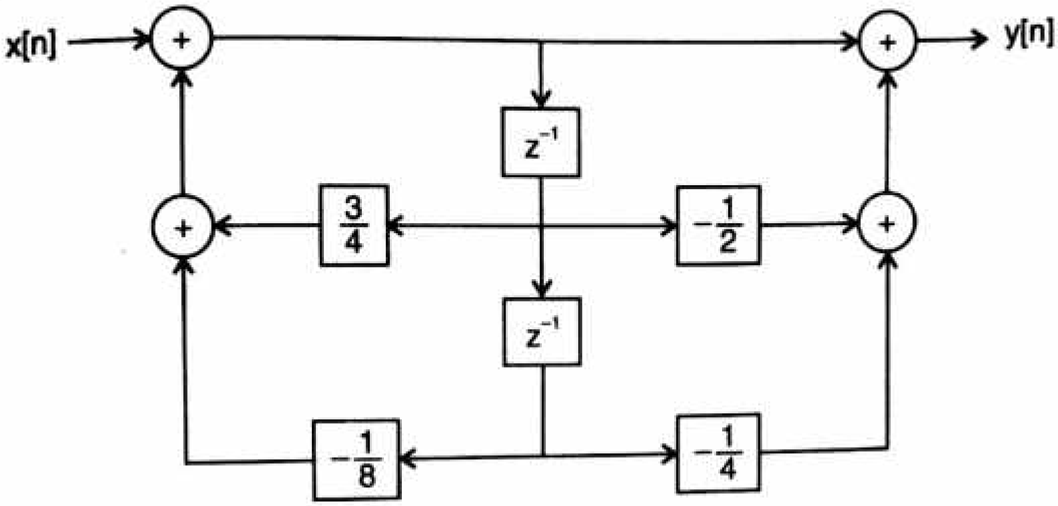

Fig. 7.17. Block-diagram representation for example 7.39

EXAMPLE 7.40. Find the z-transform of te^{-at}. *(U.P.T.U., 2005)*

Solution:
$$x(t) = te^{-at} \text{ or } x[nT] = nTe^{-anT}$$

Let
$$T = 1 \text{ then } x[n] = ne^{-\alpha n}$$

$$X(z) = \sum_{n=0}^{\infty} ne^{-\alpha n}z^{-n} = \sum_{n=0}^{\infty} n\,(e^{-\alpha}z^{-1})\,n$$

$$= \frac{e^{-\alpha}z^{-1}}{(1-e^{-\alpha}z^{-1})^2}$$

EXAMPLE 7.41. Find inverse z-transform of $\frac{2z}{(2z-1)^2}$. *(U.P.T.U., 2005)*

Solution:
$$X(z) = \frac{2z}{(2z-1)^2} = \frac{2z}{4z^2\left(1-\frac{1}{2}z^{-1}\right)^2} = \frac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

Therefore, $x[n] = n\left(\frac{1}{2}\right)^n U[n]$

EXAMPLE 7.42. Consider the system described by the difference equation:
$c(k+1) + 2c(k) = r(k);\ c(0) = 0$ *(U.P.T.U., 2005)*
Obtain the system impulse response $h(k)$.

Solution: Taking z-transform of the difference equation, we have

$$zC(z) + 2C(z) = R(z)$$

or
$$\frac{C(z)}{R(z)} = H(z) = \frac{1}{z+2}$$

Taking inverse z-transform, we have

$$h(k) = (-2)^{k-1}\, U[k-1]$$

EXAMPLE 7.43. Determine the z-transform of the following functions :

(*i*) $F(s) = \dfrac{10}{s(s^2+s+2)}$

(*ii*) $F(s) = \dfrac{2(s+1)}{s(s+5)}$ *(U.P.T.U., 2006)*

Solution: (*i*) Applying Partial fraction

$$F(s) = \frac{10}{s(s^2+s+2)}$$

$$F(s) = \frac{A}{s} + \frac{Bs+C}{(s^2+s+2)}$$

On solving for A and B we get

For A : $10 = A(s^2 + s + 2) + Bs^2 + Cs$

Put $s = 0$; we get $A = 5$

Put $s = 1$; $10 = 4A + B + C \Rightarrow B + C = -10$

Put $s = -1$; $10 = 2A + B - C \Rightarrow B - C = 0$

Simplifying we get

$$B = -5 \quad \text{and} \quad C = -5$$

or

$$F(s) = \frac{5}{s} - \frac{5(s+1)}{(s^2+s+2)}$$

$$= 5\left[\frac{1}{s} - \frac{s}{(s^2+s+2)} - \frac{1}{(s^2+s+2)}\right]$$

$$= 5\left[\frac{1}{s} - \frac{s}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2} - \frac{2}{\sqrt{7}}\,\frac{1.\frac{\sqrt{7}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right]$$

Taking Inverse Laplace transform we get

$$f(t) = 5 - 5\left[\cos\left(\frac{\sqrt{7}}{2}t\right)e^{-t/2} - \frac{10}{\sqrt{7}}\sin\left(\frac{\sqrt{7}}{2}t\right)e^{-t/2}\right]$$

$$f(t) = 5 - 5\,e^{-t/2}\left[\cos\left(\frac{\sqrt{7}}{2}t\right) + \frac{2}{\sqrt{7}}\sin\left(\frac{\sqrt{7}}{2}t\right)\right]$$

Taking z-transform

$$F(z) = \frac{5}{1-z^{-1}} - 5\left[\frac{1-e^{-1/2T}\cos\left(\frac{\sqrt{7}}{2}T\right)z^{-1}}{1-2e^{-1/2T}\cos\left(\frac{\sqrt{7}}{2}T\right)z^{-1}+e^{-2.1/2T}z^{-2}}\right] - \frac{10}{\sqrt{7}}$$

$$\left[\frac{e^{-1/2T}\sin\left(\frac{\sqrt{7}}{2}T\right)z^{-1}}{1-2e^{-1/2T}\cos\left(\frac{\sqrt{7}}{2}T\right)z^{-1}+e^{-2\,1/2T}z^{-2}}\right]$$

or
$$F(z) = 5\left[\frac{1}{1-z^{-1}}-\frac{1-e^{-1/2t}\cos\left(\frac{\sqrt{7}}{2}T\right)z^{-1}+\frac{2}{\sqrt{7}}\left(e^{-1/2T}\sin\left(\frac{\sqrt{7}}{2}T\right)z^{-1}\right)}{1-2e^{-1/2t}\cos\left(\frac{\sqrt{7}}{2}t\right)z^{-1}+e^{-t}z^{-2}}\right]$$

(*ii*)
$$F(s) = \frac{2(s+1)}{2(s+5)}$$

Applying Partial fractions

$$= \frac{A}{s} + \frac{B}{s+5}$$

Solving for A and B we get

$$F(s) = \frac{2}{5s} + \frac{8}{5(s+5)}$$

Taking inverse Laplace transform we get

$$f(t) = \frac{2}{5} + \frac{8}{5}\,e^{-5t}$$

Now taking z transform

$$F(z) = \frac{2}{5}\,\frac{1}{1-z^{-1}} + \frac{8}{5\,(1-e^{-5T}z^{-1})}.$$

EXAMPLE 7.44. Discuss the significance of the difference equation. Solve the following difference equation using the z-transform method.

$$c\,(k+2) - 0.1\,c\,(k+1) - 0.2\,c(k) = r(k+1) + r(k)$$

Where, $r(k) = 1(k)$ **for** $k = 0, 1, 2, \ldots;$
$C(0) = 0,$ **and** $C(1) = 0.$ *(U.P.T.U., 2006)*

Solution: Significance of Difference Equation: Just as differential equation can be used to represent the continuous time system $y(t)$, the difference equations can be used in to represent the discrete-time system $y(nt)$ or $y[n]$.

$$c(k+2) - 0.1\,c(k+1) - 0.2\,c(k) = r(k+1) + r(k) \quad \ldots(i)$$

(A.) $r(k)$ = unit step function
$C(0) = C(1) = 0$

Taking z-transform of equation (*i*)

$$z^2\,\{\bar{C}(z) - C_0 - C_1 z^{-1}\} - 0.1\,z\{\bar{C}(z) - C_0\} - 0.2\,\bar{C}(z) = z\cdot\frac{z}{z-1} + \frac{z}{z-1}$$

$$\bar{C}(z)\,\{z^2 - 0.1z - 0.2\} = \frac{z}{(z-1)} + \frac{z^2}{(z-1)}$$

$$\bar{C}(z) = \frac{z}{(z-1)\,(z^2-0.1z-0.2)} + \frac{z^2}{(z-1)\,(z^2-0.1z-0.2)}$$

$$\frac{\bar{C}(z)}{z} = \frac{10}{(z-1)(10z^2-z-2)} + \frac{10z}{(z-1)(10z^2-z-2)}$$

$$\frac{\bar{C}(z)}{z} = \frac{10}{(z-1)(2z-1)(5z+2)} + \frac{10z}{(z-1)(2z-1)(5z+2)}$$

Using partial fraction expansion

$$\frac{\bar{C}(z)}{z} = 10\left(\frac{1}{7(z-1)} - \frac{4}{9(2z-1)} + \frac{25}{63(5z+2)}\right) + 10\left(\frac{1}{7(z-1)} - \frac{2}{9(2z-1)} - \frac{10}{63(5z+2)}\right)$$

On solving

$$\bar{C}(z) = \frac{20}{7}\frac{z}{z-1} - \frac{30}{9}\frac{z}{(z-1/2)} + \frac{30}{63}\frac{z}{(z+2/5)}$$

Taking inverse z-transform, we have

$$c[k] = \left[\frac{20}{7} - \frac{10}{3}\left(\frac{1}{2}\right)^k + \frac{10}{21}\left(-\frac{2}{5}\right)^k\right]U[k]$$

(B.) $r(k) =$ unit ramp function

$$z^2[\bar{C}(z) - C_0 - C_1 z^{-1}] - 0.1z[\bar{C}(z) - C_0] - 0.2\,\bar{C}(z) = \frac{z^2}{(z-1)^2} + \frac{z}{(z-1)^2}$$

$$(10z^2 - z - 2)\ \bar{C}(z) = \frac{10(z^2+z)}{(z-1)^2}$$

$$C(z) = \frac{10(z^2+z)}{(z-1)^2(2z-1)(5z+2)}$$

$$= \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{2z-1} + \frac{D}{5z-2}$$

$$= -\frac{170}{49}\cdot\frac{1}{z-1} + \frac{20}{7}\cdot\frac{1}{(z-1)^2} + \frac{20}{3}\cdot\frac{1}{2z-1} + \frac{100}{147}\cdot\frac{1}{5z+2}$$

$$= -\frac{170}{49}\cdot z^{-1}\left(\frac{z}{z-1}\right) + \frac{20}{7}\cdot z^{-1}\left(\frac{z}{(z-1)^2}\right) + \frac{10}{3}z^{-1}\left(\frac{z}{z-\frac{1}{2}}\right) + \frac{20}{147}z^{-1}\left(\frac{z}{z+\frac{2}{5}}\right)$$

Taking inverse z-transform, we have

$$c(k) = \left[-\frac{170}{49}(1)^{k-1} + \frac{20}{7}(k-1) + \frac{10}{3}\left(\frac{1}{2}\right)^{k-1} + \frac{20}{147}\left(-\frac{2}{5}\right)^{k-1}\right]U[k-1]$$

EXAMPLE 7.45. Realize the system given as

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = x[n] + 2x[n-1]$$

Using z-transform with minimum no. of delay unit, assume initial condition is zero. *(U.P.T.U., 2007)*

Solution: Taking z-transform, we have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+2z^{-1}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

The realization of the system is shown in figure 7.18.

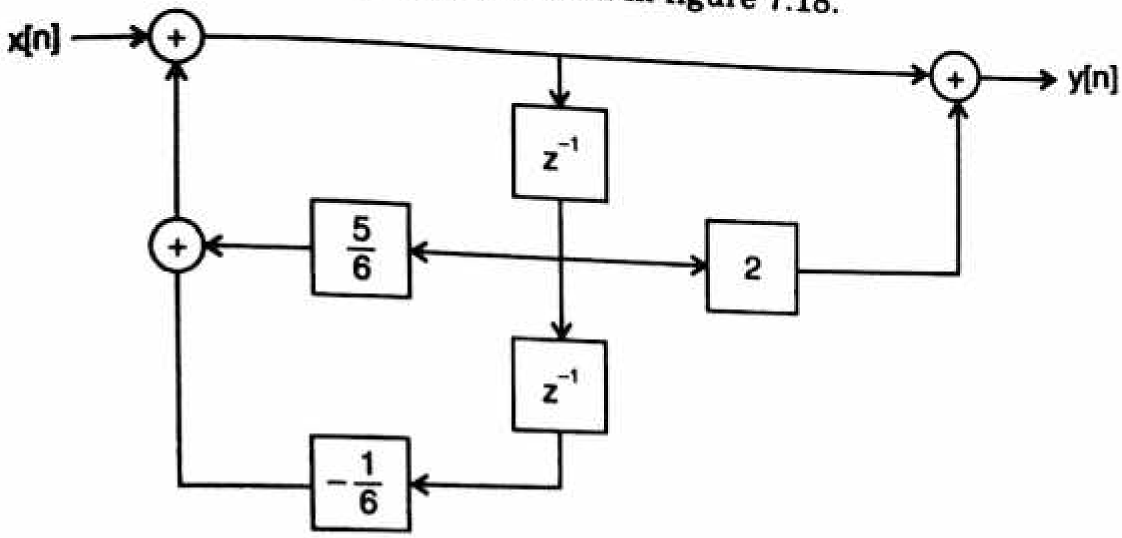

Fig. 7.18. Block diagram representation of example 7.45

EXERCISES

7.1. Define z-transform and also define region of convergence (ROC).

7.2. State and prove all properties of z-transform.

7.3. Describe the properties of ROC.

7.4. Show that if two LTI discrete-time system with impulse responses $h_1[n]$ and $h_2[n]$ are connected in cascaded or in parallel, the overall impulse response $h[n]$ of the interconnected system is $h_1[n] * h_2[n]$ or $h_1[n] + h_2[n]$, respectively.

UNSOLVED PROBLEMS

7.1. Determine $y[n]$ if $x[n] = \delta[n-1]$, and input $x[n]$ and output $y[n]$ are related by the following difference equation:

$$y[n] = \frac{1}{4}y[n-1]+x[n].$$

7.2. Find the z-transform of $f[n] = 4\delta[n-1] + 3\,U[n] + 7\,\delta[n+1] + \delta[n]$ and specify the corresponding region of convergence.

7.3. Find the z-transform of $f(nT) = e^{anT}$ and also specify the ROC.

7.4. Find the z-transform of $f(nT) = \cos \omega nT.\ U(nT)$.

7.5. Find the z-transform of $f(nT) = e^{-anT} \cos \omega nT.\ U(nT)$.

7.6. Find the z-transform of the following sequences of samples and also specify the ROC.

(a) $f[n] = \left(\frac{1}{5}\right)^n U[n-1]$

(b) $f[n] = \left(-\frac{1}{4}\right)^n U[n]$

(c) $f[n] = 2\,U[n] - 3\,U[n-5]$

(d) $f[n] = 5\,U[-n]$

7.7. Find the impulse response of the system with system function is given by